

ADAMS FILTRATION AND GENERALIZED HUREWICZ MAPS FOR INFINITE LOOPSPACES

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ABSTRACT. We study the Hurewicz map

$$h_* : \pi_*(X) \rightarrow R_*(\Omega^\infty X)$$

where $\Omega^\infty X$ is the 0th space of a spectrum X , and R_* is the generalized homology theory associated to a connective commutative S -algebra R .

We prove that the decreasing filtration of the domain associated to an R -based Adams resolution is compatible with a filtration of the range associated to the Goodwillie tower for $\Sigma^\infty \Omega^\infty X$.

One easy-to-state application goes as follows. Localize at a prime p and suppose X is $(c-1)$ -connected with $c > 0$. If $\alpha \in \pi_*(X)$ has Adams filtration s and $|\alpha| < cp^s$, then $h_*(\alpha) = 0 \in R_*(\Omega^\infty X)$.

In the special case when X is the suspension spectrum of a space Z and $R = H\mathbb{Z}/2$, we recover results announced by Lannes and Zarati in the 1980s, relating the Adams filtration of $\pi_*^S(Z)$ to Dyer-Lashof length in $H_*(QZ; \mathbb{Z}/2)$. Our methods of proof are very different from the methods outlined by them, and rely on new structure on the Topological André-Quillen ideal filtration of commutative R -algebras, applied to the algebra $R \wedge (\Omega^\infty X)_+$.

Let E_{n*} be the n th Morava E -theory. As another application of our general theory, if X is 0-connected, one gets a natural map of the Adams-Novikov spectral sequence for X to the constant spectral sequence having s -line equal to the primitives in $E_{n*}(X_{h\Sigma_{p^s}}^{\wedge p^s})$. This hints at the possibility of an algebraic ‘ E_n Singer transfer’.

1. INTRODUCTION

In algebraic topology, there is constant tension between what one would like to compute, and what one can compute. Typically one would like to compute the homotopy groups of some sort of geometric object X , while one is only able to compute the homology groups of X for some well chosen homology theories.

In stable homotopy, this paradigm often takes the following form. Let X be a spectrum, and let R_* be the homology theory associated to a ring spectrum R . The unit map $S \rightarrow R$ induces a map

$$\bar{h} : X \rightarrow R \wedge X$$

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and the induced map on homotopy groups is the Hurewicz map

$$\bar{h}_* : \pi_*(X) \rightarrow R_*(X).$$

Perhaps this map detects some part of $\pi_*(X)$, but typically there is much in the kernel.

A traditional way to deal with this kernel is via the R -based Adams Spectral Sequence. Let $X(0) = X$, and let $X(1)$ be the homotopy fiber of $X \rightarrow R \wedge X$. Then recursively define $X(s) = X(s-1)(1)$. The decreasing filtration

$$X = X(0) \leftarrow X(1) \leftarrow X(2) \leftarrow \dots$$

defines the Adams Spectral Sequence upon applying homotopy, and the original Hurewicz map appears as an edge homomorphism.

A rather different and much less studied way to ‘improve’ the Hurewicz map for X is to instead use the Hurewicz map for the zero space $\Omega^\infty X$: this is the map on homotopy groups induced by the map of spaces

$$h : \Omega^\infty X \rightarrow \Omega^\infty(R \wedge \Omega^\infty X).$$

Since π_* commutes with Ω^∞ in non-negative degrees, if X is connective (i.e., -1 -connected), this space level Hurewicz map has the form

$$h_* : \pi_*(X) \rightarrow R_*(\Omega^\infty X),$$

and refines the map \bar{h}_* , potentially significantly: the diagram

$$\begin{array}{ccc} & & R_*(\Omega^\infty X) \\ & \nearrow h_* & \downarrow \epsilon_* \\ \pi_*(X) & \xrightarrow{\bar{h}_*} & R_*(X) \end{array}$$

commutes, where $\epsilon : \Sigma^\infty \Omega^\infty X \rightarrow X$ is the counit of the adjunction $(\Sigma^\infty, \Omega^\infty)$.

The purpose of this paper is to show that, via h_* , and with R upgraded to being an E_∞ ring spectrum, the R -based Adams filtration on $\pi_*(X)$ is compatible (in an ‘exponential’ sort of way, to be described) with a natural decreasing filtration on $R_*(\Omega^\infty X)$.

The filtration on $R_*(\Omega^\infty X)$ arises most simply from the Goodwillie tower of the functor $X \mapsto \Sigma^\infty \Omega^\infty X$. But to prove our main theorem, it will be crucial to view $R \wedge (\Omega^\infty X)_+$ as a commutative augmented R -algebra, and then use the associated ‘augmentation ideal’ filtration of this algebra, in the sense of Topological André-Quillen theory. If X is connective, then these two filtrations will agree.

A key tool in our construction is a new ‘composition’ property of this augmentation ideal filtration, developed by the author with Luis Pereira [KP14]. This seems to be structure not previously exploited in the TAQ literature.

Acknowledgements The inspiration for our main theorem comes from old results of Lannes and Zarati as announced in [LZ83]: see Remark 1.8.

Early versions of our main results were announced in the September, 2007 algebraic topology workshop in Oberwolfach [K07], and conversations with Jean Lannes at that time were useful. Conversations on aspects of this work with Greg Arone, Mike Hill, Luis Pereira, Andrew Blumberg, and Mike Mandell have been helpful.

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1.1. The main theorem. As a functor of X , $\Sigma^\infty \Omega^\infty X$ admits a map to its Goodwillie tower

$$X = P_1(X) \leftarrow P_2(X) \leftarrow P_3(X) \leftarrow \dots,$$

satisfying the following basic properties: the first map $\Sigma^\infty \Omega^\infty X \rightarrow X$ is ϵ , the k th map $\Sigma^\infty \Omega^\infty X \rightarrow P_k(X)$ is k -connected if X is 0-connected, and the homotopy fiber of $P_k(X) \rightarrow P_{k-1}(X)$ is equivalent to $D_k X = (X^{\wedge k})_{h\Sigma_k}$, the k th extended power of X . (See [Goo03] and [AK02].)

We focus on the fibers of the maps to the tower. We let $I(X) = \Sigma^\infty \Omega^\infty X$, and, for $k > 1$, we let $I^k(X)$ be the homotopy fiber of $\Sigma^\infty \Omega^\infty X \rightarrow P_{k-1}(X)$. These define a decreasing filtration

$$\Sigma^\infty \Omega^\infty X = I(X) \leftarrow I^2(X) \leftarrow I^3(X) \leftarrow \dots$$

whose holimit is null if X is 0-connected, and such that the homotopy cofiber of $I^{k+1}(X) \rightarrow I^k(X)$ is $D_k X$.

Applying Ω^∞ to the R -based Adams filtration for X gives a decreasing filtration of $\Omega^\infty X$. Applying $\Omega^\infty(R \wedge -)$ to the Goodwillie tower filtration of $I(X) = \Sigma^\infty \Omega^\infty X$ gives a decreasing filtration of $\Omega^\infty(R \wedge \Omega^\infty X)$. Our main theorem says that, under mild connectivity hypotheses, these are compatible.

Theorem 1.1. *Let R be an E_∞ ring spectrum such that the cofiber of the unit, R/S , is 0-connected. Fix $d > 1$. Localized away from $(d-1)!$ and restricted to connective spectra X , the Hurewicz map*

$$h : \Omega^\infty X \longrightarrow \Omega^\infty(R \wedge \Omega^\infty X)$$

induces a natural map of filtrations

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\Omega^\infty X(3) & \xrightarrow{h_3} & \Omega^\infty(R \wedge I^{d^3}(X)) \\
\downarrow & & \downarrow \\
\Omega^\infty X(2) & \xrightarrow{h_2} & \Omega^\infty(R \wedge I^{d^2}(X)) \\
\downarrow & & \downarrow \\
\Omega^\infty X(1) & \xrightarrow{h_1} & \Omega^\infty(R \wedge I^d(X)) \\
\downarrow & & \downarrow \\
\Omega^\infty X(0) & \xrightarrow{h_0=h} & \Omega^\infty(R \wedge I(X)).
\end{array}$$

The connectivity hypothesis on R ensures that $X(s) \simeq X \wedge (\Sigma^{-1}R/S)^{\wedge s}$ is connective for all s , if X is. Note that if $d = 2$, no localization is needed.

Important ring spectra R satisfying the hypotheses of the theorem include MU , ku , ko , tmf , $H\mathbb{Z}$, and $H\mathbb{Z}/p$, with p a prime.

Remark 1.2. We have the following useful addendum. All of our functors are suitably enriched over topological spaces (or simplicial sets) and our natural transformations are suitably continuous, so one gets compatibility under suspension in the X variable: there are natural transformations of filtrations $\Delta : \Sigma I^d(X) \rightarrow I(\Sigma^d X)$, and the diagrams

$$\begin{array}{ccc}
\Omega^\infty X(s) & \xrightarrow{h_s(X)} & \Omega^\infty(R \wedge I^{d^s}(X)) \\
\parallel & & \parallel \\
& & \Omega\Omega^\infty(R \wedge \Sigma I^{d^s}(X)) \\
& & \downarrow \Delta \\
\Omega\Omega^\infty(\Sigma X)(s) & \xrightarrow{\Omega h_s(\Sigma X)} & \Omega\Omega^\infty(R \wedge I^{d^s}(\Sigma X))
\end{array}$$

commute. Thus the bottom horizontal map factors through the map Δ .

This is a significant constraint. Standard arguments show that, localized at a prime p , $\Delta : \Sigma D_r X \rightarrow D_r \Sigma X$ is null unless r is a power of p . We deduce the following.

Addendum 1.3. *Under the hypotheses of Theorem 1.1, if X is 0-connected, the composite*

$$\Omega\Omega^\infty X(s) \xrightarrow{\Omega h_s} \Omega\Omega^\infty(R \wedge I_{d^s}(X)) \rightarrow \Omega\Omega^\infty(R \wedge D_{d^s} X)$$

is null unless d is prime, and, in this case, this map factors through

$$\Delta : \Omega\Omega^\infty(R \wedge \Sigma D_{d^s} \Sigma^{-1} X) \rightarrow \Omega\Omega^\infty(R \wedge D_{d^s} X).$$

So what are the computational implications of our main theorem?

Most obviously, if one applies π_* to the map of towers, one gets a map of spectral sequences from the R -based Adams spectral sequence converging to $\pi_*(X)$, to the ‘ d -exponential’ Goodwillie tower spectral sequence converging to $R_*(\Omega^\infty X)$, and with E^1 s -line equal to $R_*(I^{d^s}(X)/I^{d^{s+1}}(X))$. Moreover, if R maps to another spectrum B (typically an R -algebra of some sort), then one can compose our map of towers (and thus spectral sequences) with the evident map of towers covering $\Omega^\infty(R \wedge \Omega^\infty X) \rightarrow \Omega^\infty(B \wedge \Omega^\infty X)$.

In particular, if one lets $F_s \pi_*(X) = \text{Im}\{\pi_*(X(s)) \rightarrow \pi_*(X)\}$, we have the next tidy corollary.

Corollary 1.4. *Let R be as in the Theorem 1.1. If X is connective, then, localized away from $(d-1)!$, the composite*

$$F_s \pi_*(X) \subseteq \pi_*(X) \xrightarrow{h_*} R_*(\Omega^\infty X) \rightarrow R_*(P_k(X))$$

is zero if $k < d^s$, and, if $k = d^s$, this composite factors through

$$R_*(D_{d^s} X) \rightarrow R_*(P_{d^s}(X)).$$

Furthermore, if X is 0-connected, this last composite factors through

$$\Delta_* : R_{*-1}(D_{d^s} \Sigma^{-1} X) \rightarrow R_*(D_{d^s} X),$$

and is thus zero unless d is a prime.

A variant of this corollary, focused on connectivity, goes as follows.

Corollary 1.5. *Let R be as in the Theorem 1.1, and localize at a prime p . Suppose X is $c-1$ -connected for some $c \geq 1$. If $\alpha \in \pi_*(X)$ has R -based Adams filtration s and $|\alpha| < cp^s$, then $h_*(\alpha) = 0 \in R_*(\Omega^\infty X)$.*

This follows immediately from the main theorem: if X is $(c-1)$ -connected, then $I^k(X)$ is $(kc-1)$ -connected, and thus the Hurewicz image of an element α of Adams filtration s will factor through a space that is (cp^s-1) -connected.

Corollary 1.5 says that $\ker\{h_* : \pi_*(X) \rightarrow R_*(\Omega^\infty X)\}$ contains everything above a logarithmic curve, when $\pi_*(X)$ is displayed with standard Adams spectral sequence charts having horizontal axis $|\alpha|$ and vertical axis the Adams filtration of α .

Sometimes there is very little left under this curve. This will be the case in the family of examples studied in §5, where we will show that the corollary allows one to easily determine the mod 2 Hurewicz map for BO and any of its connective covers. In particular, it correctly predicts that

$$h_* : \pi_*(bo) \rightarrow H_*(BO; \mathbb{Z}/2)$$

has nonzero image precisely in degrees 1, 2, 4, and 8 corresponding to elements in $\pi_*(bo)$ of Adams filtration 0, 1, 2, and 3. This reproves Milnor’s

theorem [Mil58] that S^n admits a vector bundle ξ with $w_n(\xi) \neq 0$ only when $n = 1, 2, 4, 8$.

The author is currently exploring the application of the corollary to other interesting situations. For example, with a little bit of cleverness, one can also quite easily show that

$$h_* : \pi_*(tmf) \rightarrow H_*(\Omega^\infty tmf; \mathbb{Z}/2)$$

is nonzero in precisely dimensions 1,2,3,6,8. (Curiously, the range here, $H_*(\Omega^\infty tmf; \mathbb{Z}/2)$, has yet to be completely calculated.) Details will appear elsewhere.

Our main theorem has yet more compelling consequences in cases when the spectral sequences associated to the Goodwillie tower collapses. We now discuss two such situations.

1.2. Consequences when X is a suspension spectrum. When $X = \Sigma^\infty Z$, with Z a based space, our Hurewicz map has the form

$$h_* : \pi_*^S(Z) \longrightarrow R_*(QZ),$$

where $QZ = \Omega^\infty \Sigma^\infty Z$, as usual.

In this case, the filtration for $\Sigma^\infty QZ$ splits; more precisely, as described in [K06], there are natural maps of filtered spectra

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \bigvee_{k=3}^\infty D_k(Z) & \longrightarrow & I^3(Z) & \longrightarrow & \prod_{k=3}^\infty D_k(Z) \\ \downarrow & & \downarrow & & \downarrow \\ \bigvee_{k=2}^\infty D_k(Z) & \longrightarrow & I^2(Z) & \longrightarrow & \prod_{k=2}^\infty D_k(Z) \\ \downarrow & & \downarrow & & \downarrow \\ \bigvee_{k=1}^\infty D_k(Z) & \longrightarrow & \Sigma^\infty QZ & \longrightarrow & \prod_{k=1}^\infty D_k(Z), \end{array}$$

with the horizontal composition being the evident map, inducing equivalences on filtration cofibers at all levels. (The naturality is with respect to actual maps between spaces, not stable maps.) If Z is also path connected, all horizontal maps are equivalences, and one gets equivalences of strongly convergent towers. (We have suppressed many Σ^∞ 's from the notation.)

Let $E_{*,*}^r(Z)$ denote the r th page of R -based Adams spectral sequence converging to $\pi_*^S(Z)$, with cycles and boundaries $Z^r(Z)$ and $B^r(Z)$. Since the spectral sequence for the tower of QZ collapses, we conclude the following.

Theorem 1.6. *Let R be as in Theorem 1.1. Localized at a prime p , the Hurewicz map*

$$h_* : \pi_*^S(Z) \longrightarrow R_*(QZ)$$

induces maps

$$h_{s,t}^r : E_{s,t}^r(Z) \rightarrow \bigoplus_{k=p^s}^{p^{s+1}-1} R_{t-s}(D_k(Z))$$

all of which factor through a common map

$$h_{s,t} : Z_{s,t}^1(Z)/B_{s,t}^\infty(Z) \rightarrow \bigoplus_{k=p^s}^{p^{s+1}-1} R_{t-s}(D_k(Z)).$$

Furthermore, if $Z \simeq \Sigma W$ for some space W , the image of $h_{s,t}$ is contained in the image of $\Delta_* : R_{t-s-1}(D_{p^s}(W)) \rightarrow R_{t-s}(D_{p^s}(Z))$.

In particular, there is a natural diagram

$$\begin{array}{ccc} E_{s,t}^2(Z) & \xrightarrow{h_{s,t}^2} & \bigoplus_{k=p^s}^{p^{s+1}-1} R_{t-s}(D_k(Z)), \\ \downarrow & \searrow h_{s,t} & \\ Z_{s,t}^2(Z)/B_{s,t}^\infty(Z) & \xrightarrow{h_{s,t}} & \\ \uparrow & \nearrow h_{s,t}^\infty & \\ E_{s,t}^\infty(Z) & & \end{array}$$

with $\text{Im } h_{s,t} \subseteq \text{Im } \Delta_* \subseteq R_{t-s}(D_{p^s}(Z))$ if Z is a suspension.

We consider what this theorem says when $R = H\mathbb{Z}/p$. Write $H^*(Z)$ and $H_*(Z)$ for the mod p cohomology and homology of Z , which are left and right modules over the mod p Steenrod algebra \mathcal{A} . In this case, there is a natural isomorphism

$$\mathcal{R}_s H_*(Z) \simeq \text{Im } \Delta_* \subseteq H_*(D_{p^s}(Z)),$$

where \mathcal{R}_s is a well known endofunctor of the category of locally finite right \mathcal{A} -modules: roughly put, $\mathcal{R}_s M$ is the module generated by applying all sequences of Dyer-Lashof operations of length s to a module M . (See [Po11, KM13] for modern presentations of the interesting properties of these ‘Singer functors’.)

We let $\mathcal{R}M = \bigoplus_{t=0}^\infty \mathcal{R}_t M$. This has a decreasing filtration with $F_s \mathcal{R}M = \bigoplus_{t=s}^\infty \mathcal{R}_t M$. If Z is a suspension, one has a canonical inclusion

$$\mathcal{R}H_*(Z) \subset H_*(QZ)$$

which identifies with the module of primitives in $H_*(QZ)$, viewed as a Hopf algebra. Thus the Hurewicz map for QZ factors

$$\pi_*^S(Z) \longrightarrow \mathcal{R}H_*(Z) \subset H_*(QZ).$$

Theorem 1.6 tells us the following.

Corollary 1.7. *If a space Z is a suspension, the mod p Hurewicz map*

$$h_* : \pi_*^S(Z) \longrightarrow \mathcal{R}H_*(Z) \subset H_*(QZ)$$

is filtration preserving and induces commutative diagrams

$$\begin{array}{ccc}
 \text{Ext}_{\mathcal{A}}^{s,t}(H^*(Z), \mathbb{Z}/p) & & \\
 \downarrow & \searrow h_{s,t}^2 & \\
 Z_{s,t}^2(Z)/B_{s,t}^\infty(Z) & \xrightarrow{h_{s,t}} & \mathcal{R}_s H_*(Z) \subset H_*(D_{p^s} Z). \\
 \uparrow & \nearrow h_{s,t}^\infty & \\
 E_{s,t}^\infty(Z) & &
 \end{array}$$

Remark 1.8. Without calculation, we have thus recovered, and extended to odd primes, slight variants of the 2-primary results of Lannes and Zarati announced in [LZ83]. ([LZ] is a partially completed manuscript. See also [L88, LZ87].) Though we don't do this here, Lannes and Zarati's work suggests that it should be possible to identify $h_{s,t}^2$ as the specialization of an explicit algebraic natural transformation (of Singer transfer type)

$$\text{Ext}_{\mathcal{A}}^{s,t}(M^\vee, \mathbb{Z}/2) \rightarrow (\mathcal{R}_s M)_{t-s},$$

where M^\vee denotes the dual of a right \mathcal{A} -module M .

Though our methods are very different than theirs, the idea of the author that these old results could be naturally placed within the context of Goodwillie calculus was the starting point of this paper.

1.3. Consequences for the Hurewicz map in Morava E -theory. Let E_n denote the n th Morava E -theory at a fixed prime p , and then let $E_*(X) = \pi_*(L_{K(n)}(E_n \wedge X))$. There is a natural map

$$MU \wedge Y \rightarrow L_{K(n)}(E_n \wedge Y)$$

for all spectra Y , and thus a natural map of filtrations

$$MU \wedge I^d(X) \rightarrow L_{K(n)}(E_n \wedge I^d(X))$$

for all spectra X .

The main result in [K06] implies that the filtration for $L_{K(n)}(\Sigma^\infty \Omega^\infty X)$ has a natural splitting analogous to the splitting of the filtration for $\Sigma^\infty QZ$ as described above.

Just as before, this splitting and Theorem 1.1 show that a map of spectral sequences on the E_2 -page determines the map on the E_∞ -page. In the next theorem, let $E_{*,*}^r(X)$ denote the r th page of Adams–Novikov spectral sequence converging to $\pi_*(X)$, with cycles and boundaries $Z^r(X)$ and $B^r(X)$.

Theorem 1.9. *Fix a prime p and n . For connective spectra X , the n th Morava E -theory Hurewicz map*

$$h_* : \pi_*(X) \longrightarrow E_*(\Omega^\infty X)$$

induces maps

$$h_{s,t}^r : E_{s,t}^r(X) \rightarrow \bigoplus_{k=p^s}^{p^{s+1}-1} E_{t-s}(D_k(X))$$

all of which factor through a common map

$$h_{s,t} : Z_{s,t}^1(X)/B_{s,t}^\infty(X) \rightarrow \bigoplus_{k=p^s}^{p^{s+1}-1} E_{t-s}(D_k(X)).$$

Furthermore, if X is 0-connected, the image of $h_{s,t}$ is contained in the image of $\Delta_* : E_{t-s-1}(D_{p^s}(\Sigma^{-1}X)) \rightarrow E_{t-s}(D_{p^s}(X))$.

In particular, there is a natural diagram

$$\begin{array}{ccc} \text{Ext}_{MU_*(MU)}^{s,t}(MU_*, MU_*(X)) & & \\ \downarrow & \searrow h_{s,t}^2 & \\ Z_{s,t}^2(Z)/B_{s,t}^\infty(X) & \xrightarrow{h_{s,t}} & \bigoplus_{k=p^s}^{p^{s+1}-1} E_{t-s}(D_k(X)), \\ \uparrow & \nearrow h_{s,t}^\infty & \\ E_{s,t}^\infty(X) & & \end{array}$$

with $\text{Im } h_{s,t} \subseteq \text{Im } \Delta_* \subseteq E_{t-s}(D_{p^s}(X))$ if X is 0-connected.

Remarks 1.10. (a) If $E_*(X)$ is a finitely generated free E_* -module, there is much known about $\text{Im } \Delta_* \subset E_*(D_{p^s}(X))$; in particular, it is a functor of $E_*(X)$, viewed as an E_* -module [Rez09, Remark 7.4]. One might expect that the map $h_{*,*}^2$ has an algebraic description in this case.

(b) As E is not bounded below, convergence of the (trivial) tower spectral sequence is both problematic and subtle. This is studied in detail in [K06]. In particular, necessary and sufficient conditions are found guaranteeing that the natural map

$$\bigoplus_{k=0}^{\infty} E_*(D_k X) \rightarrow E_*(\Omega^\infty X)$$

is an isomorphism. When $n = 1$, this is the case if X is 1-connected and $\pi_2(X)$ is torsion. When $n = 2$, this is the case if X is 2-connected, $\pi_3(X)$ is torsion, and $K(1)_*(X) = 0$. For higher n , there are ambiguities due to the unresolved Telescope Conjecture.

1.4. Organization of the proof of Theorem 1.1. To prove our main theorem, we need to regard $I(X)$ in a very different way.

Recall that an infinite loopspace $\Omega^\infty X$ is naturally a \mathcal{C}_∞ -space, where \mathcal{C}_∞ is the usual little cubes operad. The suspension spectrum $\Sigma^\infty(\Omega^\infty X)_+$ then inherits the structure of a \mathcal{C}_∞ -algebra in S -modules, with natural augmentation $\Sigma^\infty(\Omega^\infty X)_+ \rightarrow S$. (Readers are welcome to replace \mathcal{C}_∞ by

their favorite E_∞ operad – e.g., the linear isometries operad \mathcal{L} – in the discussion that follows.)

We redefine $I(X)$ to be the homotopy fiber of this augmentation, and regard I as a functor from S -modules (i.e., spectra) to \mathcal{C} -algebras, where \mathcal{C} is the reduced operad in pointed spaces, with $\mathcal{C}(0) = *$, $\mathcal{C}(1) = S^0$, and $\mathcal{C}(k) = \mathcal{C}_\infty(k)_+$ for $k \geq 2$. Note that we have the correct homotopy type, as the composite

$$I(X) \rightarrow \Sigma^\infty(\Omega^\infty X)_+ \rightarrow \Sigma^\infty \Omega^\infty X$$

is clearly an equivalence.

Our filtration of $I(X)$, for X connective, then is just a special case of the natural ‘Topological André–Quillen ideal filtration’ of any \mathcal{C} -algebra I in R -modules: a filtration of the form

$$\dots \rightarrow I^4 \rightarrow I^3 \rightarrow \dots \rightarrow I^2 \rightarrow I^1 = I.$$

This is a variant of constructions in the literature [McMi04, K06, HaHe13], but very critical to us is one property that seems to have not been noticed before: the operad structure of \mathcal{C} induces natural pairings

$$(I^j)^i \rightarrow I^{ij}$$

In more generality, these sorts of constructions and pairings will be developed by Luis Pereira and the author in [KP14]. Definitions and properties will be reviewed in §2.

Using these properties, in §3, we will show how Theorem 1.1 follows quickly from the construction of natural liftings of \mathcal{C} -algebras in R -modules

$$\begin{array}{ccc} & & R \wedge I^d(X) \\ & \nearrow \text{dashed} & \downarrow \\ R \wedge I(X(1)) & \longrightarrow & R \wedge I(X). \end{array}$$

when localized away from $(d-1)!$. Constructing these when $d = 2$ will be quite formal, using properties of André–Quillen homology. For larger d , an extra argument is needed, a bit delicate and using both André–Quillen theory and Goodwillie calculus: see §4.

In §5, we illustrate our results by computing the mod 2 Hurewicz map for all connective covers of BO , starting from a minimal amount of stable information about ko .

2. THE TOPOLOGICAL ANDRÉ–QUILLEN IDEAL FILTRATION

We work throughout with one of the standard, and Quillen equivalent, modern symmetric monoidal categories of S -modules modeling spectra as in [MMSS01].

If R is a commutative S -algebra, we let Mod_R denote the category of R -modules. If \mathcal{C} is then an operad as in the last section, we let $\text{Mod}_R[\mathcal{C}]$ denote the category of \mathcal{C} -algebras in R -modules. Both Mod_R and $\text{Mod}_R[\mathcal{C}]$ are model categories in which weak equivalences and fibrations are determined

by viewing the maps as being in Mod_S . (See [Ha09] for this result using the S -module category of [Sh04], and [CM13, §7] for a general result of this type.)

In this section we review certain natural constructions on $\text{Mod}_R[\mathcal{C}]$. Versions of most of this are in the literature (e.g. [BM05, HaHe13]), and details of the parts that aren't will appear in [KP14].

2.1. Operadic terminology. To describe our constructions, it seems prudent to recall some basic operadic terminology.

A symmetric sequence \mathcal{M} in a category \mathcal{S} is a sequence of objects $M(k)$, where $M(k)$ is equipped with an action of the k th symmetric group Σ_k . If \mathcal{S} is symmetric monoidal there is an associative composition product $\mathcal{M} \circ \mathcal{N}$ on such sequences, and an operad \mathcal{C} is precisely a monoid with respect to this structure. The operad we are considering in this paper is nonunital, i.e., $C(0) = *$, and we also have that $C(1) = S^0$.

Given an operad \mathcal{C} , one can define left, right, and bi-modules in the evident way, and form constructions like $\mathcal{M} \circ_{\mathcal{C}} \mathcal{N}$ with expected properties. A left \mathcal{C} -module I is a \mathcal{C} -algebra if $I(k) = *$ for all $k > 0$: one writes ' I ' again for the object $I(0)$. The \mathcal{C} -algebras in \mathcal{S} form a category, which denoted by $\mathcal{S}[\mathcal{C}]$. It is easy to see that if \mathcal{M} is a \mathcal{C} -bimodule and I is a \mathcal{C} -algebra, then $\mathcal{M} \circ_{\mathcal{C}} I$ is again a \mathcal{C} -algebra.

When $\mathcal{S} = \text{Mod}_R$, and one unpacks the definitions, $I \in \text{Mod}_R[\mathcal{C}]$ is an R -module I equipped with suitably compatible maps

$$C(k) \wedge_{\Sigma_k} I^{\wedge k} \rightarrow I$$

for all $k \geq 1$.

2.2. Topological André–Quillen homology. Let $z : R\text{-mod} \rightarrow \text{Mod}_R[\mathcal{C}]$ be the functor that assigns the trivial \mathcal{C} -algebra structure to an R -module. More precisely, and since we have assumed that $C(1) = S^0$, given an R -module M , $z(M)$ is just M with trivial structure maps $C(k) \wedge_{\Sigma_k} M^{\wedge k} \rightarrow M$ for all $k \geq 2$.

Various authors, especially [B99, BM05], study the derived version of the left adjoint to z , a flavor of Topological André–Quillen homology. We recall the definition in our context [Ha10].

Let $c : \text{Mod}_R[\mathcal{C}] \rightarrow \text{Mod}_R[\mathcal{C}]$ denote cofibrant replacement.

Let S_R denote the symmetric sequence of R -modules which is R in degree 1, and $*$ otherwise. This is a right \mathcal{C} -module (actually a bi-module).

Definition 2.1. Let $Q_R : \text{Mod}_R[\mathcal{C}] \rightarrow \text{Mod}_R$ be defined by

$$Q_R(I) = S_R \circ_{\mathcal{C}} c(I).$$

Proposition 2.2. Q_R satisfies the following properties.

(a) Q_R takes homotopy pushouts in $\text{Mod}_R[\mathcal{C}]$ to homotopy pushouts in Mod_R .

(b) Q_R induces a left adjoint to z in the associated homotopy categories: there are natural isomorphisms

$$[Q_R(I), M]_{\text{Mod}_R} \simeq [I, z(M)]_{\text{Mod}_R[\mathcal{C}]}$$

for all $I \in \text{Mod}_R[\mathcal{C}]$ and $M \in \text{Mod}_R$.

(c) The natural map $R \wedge Q_S(I) \rightarrow Q_R(R \wedge I)$ is a weak equivalence for all $I \in \text{Mod}_S[\mathcal{C}]$.

We will need the following calculation from [BM05, K06].

Proposition 2.3. *Let $I(X) = \text{hofib}\{\Sigma^\infty(\Omega^\infty X)_+ \rightarrow S\}$, regarded as an object in $\text{Mod}_S[\mathcal{C}]$. For connective X , there is a natural equivalence of S -modules*

$$Q_S(I(X)) \simeq X$$

such that, under the equivalence $I(X) \rightarrow \Sigma^\infty(\Omega^\infty X)_+ \rightarrow \Sigma^\infty \Omega^\infty X$, the natural map $I(X) \rightarrow Q_S(I(X))$ identifies with $\epsilon : \Sigma^\infty \Omega^\infty X \rightarrow X$.

2.3. The ideal filtration. Intuitively, $I \in \text{Mod}_R[\mathcal{C}]$ can be viewed as a non-unital algebra, or equivalently, as the augmentation ideal of an augmented algebra. One expects to then be able to form powers of this ideal, e.g. objects $I^k \rightarrow I$ for all $k \geq 1$ with nice properties.

For $k \geq 1$, let $\mathcal{C}^{\geq k}$ denote the symmetric sequence in based spaces with j th space equal to $\mathcal{C}(j)_+$ for $j \geq k$, and $*$ otherwise. This is a \mathcal{C} -bimodule in spaces, and thus $R \wedge \mathcal{C}^{\geq k}$ is a \mathcal{C} -bimodule in Mod_R .

Definitions 2.4. (a) For $k \geq 1$, let $(\)_R^k : \text{Mod}_R[\mathcal{C}] \rightarrow \text{Mod}_R[\mathcal{C}]$ be defined by letting $I_R^k = (R \wedge \mathcal{C}^{\geq k}) \circ_{\mathcal{C}} c(I)$.

(b) Let $I_R^{k+1} \rightarrow I_R^k$ be the natural transformation induced by the evident map of \mathcal{C} -bimodules $\mathcal{C}^{\geq k+1} \rightarrow \mathcal{C}^{\geq k}$.

Theorem 2.5. *These functors and natural transformations satisfy the following properties.*

(a) $I_R^1 = I$.

(b) $I_R^{k+1} \rightarrow I_R^k$ fits into a natural homotopy fibration sequence in $\text{Mod}_R[\mathcal{C}]$:

$$I_R^{k+1} \rightarrow I_R^k \rightarrow z(D_k^R Q_R(I)),$$

where $D_k^R M$ denotes $\mathcal{C}(k) \wedge_{\Sigma_m} M^{\wedge k}$, given $M \in \text{Mod}_R$.

(c) $I_R^k \rightarrow I$ fits into a natural homotopy fibration sequence in $\text{Mod}_R[\mathcal{C}]$:

$$I_R^k \rightarrow I \rightarrow p_{k-1}(I),$$

where $p_{k-1}(I)$ denotes the $(k-1)$ st stage of the completion tower of [HaHe13].

(d) The natural map $R \wedge I_S^k \rightarrow (R \wedge I)_R^k$ is a weak equivalence in $\text{Mod}_R[\mathcal{C}]$ for all $I \in \text{Mod}_S[\mathcal{C}]$.

2.4. Composition structure on the ideal filtration.

Definition 2.6. Let $\mu_{i,j} : (I_R^j)_R^i \rightarrow I_R^{ij}$ be the natural transformation induced by the map of \mathcal{C} -bimodules

$$\mathcal{C}^{\geq i} \circ_{\mathcal{C}} \mathcal{C}^{\geq j} \rightarrow \mathcal{C}^{\geq ij}$$

coming from the structure maps of \mathcal{C} .

Theorem 2.7. The natural transformations $\mu_{i,j} : (I_R^j)_R^i \rightarrow I_R^{ij}$ satisfy the following properties.

(a) For all k , $\mu_{k,1} = \mu_{1,k} : I_R^k \rightarrow I_R^k$ is the identity map.

(b) The $\mu_{i,j}$ are compatible as i and j vary, and the natural diagram

$$\begin{array}{ccc} (I_R^j)_R^i & \xrightarrow{\mu_{i,j}} & I_R^{ij} \\ \downarrow & & \downarrow \\ z(D_i^R D_j^R Q_R(I)) & \xrightarrow{z(\mu)} & z(D_{ij}^R Q_R(I)) \end{array}$$

commutes, where $\mu : D_i^R D_j^R M \rightarrow D_{ij}^R M$ is the map induced from the operad composition structure map

$$C(i) \times_{\Sigma_i} C(j)^i \rightarrow C(ij).$$

(c) For all $I \in \text{Mod}_S[\mathcal{C}]$, and S -algebras R , the diagram

$$\begin{array}{ccc} R \wedge (I_S^j)_S^i & \xrightarrow{1_R \wedge \mu_{i,j}} & R \wedge I_S^{ij} \\ \downarrow \wr & & \downarrow \wr \\ ((R \wedge I)_R^j)_R^i & \xrightarrow{\mu_{i,j}} & (R \wedge I)_R^{ij} \end{array}$$

commutes.

3. PROOF OF THEOREM 1.1

Recall our hypotheses: R/S is 0-connected, and X is a connective S -module.

It follows that each of the S -modules $X(s)$ is also connective. As one consequence, for all s , the canonical maps of S -modules $I(X(s)) \rightarrow Q_S(I(X(s)))$ identify with the maps $\Sigma^\infty \Omega^\infty X(s) \rightarrow X(s)$.

By slight abuse of notation, we let $I^k(X)$ denote $(I(X))_S^k$. This is consistent with the notation in the introduction: there are fibrations of \mathcal{C} -algebras

$$I^{k+1}(X) \rightarrow I^k(X) \rightarrow z(D_k(X)),$$

and the fibration sequence

$$I^k(X) \rightarrow I(X) \rightarrow p_{k-1}(I(X))$$

of Theorem 2.5 identifies with the fibration sequence

$$I^k(X) \rightarrow \Sigma^\infty \Omega^\infty X \rightarrow P_{k-1}(X)$$

coming from the Goodwillie tower for $\Sigma^\infty \Omega^\infty$.

More generally, let $I_R^k(X)$ denote $(R \wedge I(X))_R^k \simeq R \wedge I^k(X)$.

Armed with the composition structure in our ideal filtrations, Theorem 1.1 follows quite quickly from the next proposition.

Proposition 3.1. *Localized away from $(d-1)!$, there exists a natural lifting in $\text{Mod}_R[\mathcal{C}]$,*

$$\begin{array}{ccc} & & I_R^d(X) \\ & \nearrow j & \downarrow \\ I_R(X(1)) & \longrightarrow & I_R(X), \end{array}$$

where the bottom map is induced by $X(1) \rightarrow X$.

Postponing the proof until the next section, we show how Theorem 1.1 follows. The key step is the following corollary of the proposition.

Corollary 3.2. *Localized away from $(d-1)!$, for all $k \geq 1$, j induces a natural liftings in $\text{Mod}_R[\mathcal{C}]$:*

$$\begin{array}{ccc} & & I_R^{dk}(X) \\ & \nearrow j_k & \downarrow \\ I_R^k(X(1)) & \longrightarrow & I_R^k(X). \end{array}$$

Furthermore, the lifts are compatible as k varies: the diagram

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ I_R^3(X(1)) & \xrightarrow{j_3} & I_R^{3d}(X) \\ \downarrow & & \downarrow \\ I_R^2(X(1)) & \xrightarrow{j_2} & I_R^{2d}(X) \\ \downarrow & & \downarrow \\ I_R(X(1)) & \xrightarrow{j_1=j} & I_R^d(X) \end{array}$$

commutes.

Proof. The map j_k is given as the composite

$$I_R^k(X(1)) \xrightarrow{I_R^k(j)} I_R^k(I_R^d(X)) \xrightarrow{\mu_{k,d}} I_R^{dk}(X).$$

□

Proof of Theorem 1.1. Let $j(s) : I_R(X(s)) \rightarrow I_R^{ds}(X)$ be the composite

$$I_R(X(s)) \xrightarrow{j} I_R^d(X(s-1)) \xrightarrow{j_d} \dots \xrightarrow{j_{ds-2}} I_R^{ds-1}(X(1)) \xrightarrow{j_{ds-1}} I_R^{ds}(X).$$

By construction, these fit into a commutative diagram in $\text{Mod}_R[\mathcal{C}]$,

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ I_R(X(2)) & \xrightarrow{j(2)} & I_R^{d^2}(X) \\ \downarrow & & \downarrow \\ I_R(X(1)) & \xrightarrow{j(1)} & I_R^d(X) \\ \downarrow & & \downarrow \\ I_R(X(0)) & \xrightarrow{j(0)=id} & I_R(X). \end{array}$$

Precomposing with the natural maps $I(X(s)) \rightarrow I_R(X(s))$ gives a commutative diagram in $\text{Mod}_S[\mathcal{C}]$,

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ I(X(2)) & \longrightarrow & I_R^{d^2}(X) \\ \downarrow & & \downarrow \\ I(X(1)) & \longrightarrow & I_R^d(X) \\ \downarrow & & \downarrow \\ I(X(0)) & \longrightarrow & I_R(X). \end{array}$$

Forgetting algebra structure, and recalling that $I(X) \simeq \Sigma^\infty \Omega^\infty X$ and $I_R^k(X) \simeq R \wedge I^k(X)$, we have a commutative diagram of S -modules

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \Sigma^\infty \Omega^\infty X(2) & \longrightarrow & R \wedge I^{d^2}(X) \\
 \downarrow & & \downarrow \\
 \Sigma^\infty \Omega^\infty X(1) & \longrightarrow & R \wedge I^d(X) \\
 \downarrow & & \downarrow \\
 \Sigma^\infty \Omega^\infty X(0) & \longrightarrow & R \wedge I(X).
 \end{array}$$

The adjoints of these horizontal maps define natural maps of spaces

$$h_s : \Omega^\infty X(s) \rightarrow \Omega^\infty (R \wedge I^{d^s}(X))$$

fitting into the diagram of the theorem. \square

4. PROOF OF PROPOSITION 3.1

Recall that there are fibration sequences in $\mathcal{C}_R\text{-alg}$:

$$I_R^{k+1}(X) \rightarrow I_R^k(X) \rightarrow z(R \wedge D_k X).$$

Thus the obstructions to finding a natural lifting in $\mathcal{C}_R\text{-alg}$,

$$\begin{array}{ccc}
 & & I_R^d(X) \\
 & \nearrow & \downarrow \\
 I_R(X(1)) & \longrightarrow & I_R(X),
 \end{array}$$

will be natural elements $o_k(X)$, for $k = 1, \dots, d-1$, in

$$\begin{aligned}
 [I_R(X(1)), z(R \wedge D_k X)]_{\mathcal{C}_R\text{-alg}} &\simeq [R \wedge X(1), R \wedge D_k X]_{R\text{-mod}} \\
 &\simeq [X(1), R \wedge D_k X] \\
 &\simeq [S(1) \wedge X, R \wedge D_k X],
 \end{aligned}$$

where $S(1)$ is the homotopy fiber of the unit map $\eta : S \rightarrow R$.

Lemma 4.1. $o_1(X)$ is zero.

Proof. $o_1(X)$ identifies as the composite $S(1) \wedge X \rightarrow S \wedge X \rightarrow R \wedge X$, and so is naturally null. \square

This proves Proposition 3.1, when $d = 2$.

For $2 \leq k \leq d-1$ we use a rather different argument to show that $o_k(X)$ is zero, after localizing away from $k!$.

Firstly, transfer arguments show that $X^{\wedge k} \rightarrow D_k X$ is naturally split epic, localized away from $k!$. Thus we can regard $o_k(X)$ as being a natural element in $[S(1) \wedge X, R[\frac{1}{k!}] \wedge X^{\wedge k}]$.

It is important to note that the naturality is with respect to the category of *connective* S -modules X . Equivalently, we can regard $o_k(X)$ as being a natural element in $[S(1) \wedge X\langle 0 \rangle, R[\frac{1}{k!}] \wedge X\langle 0 \rangle^{\wedge k}]$, where now naturality is with respect to the category of *all* S -modules X . (Here we have used the notational convention: $X\langle c \rangle$ denotes the $(c-1)$ -connected cover of X .)

Thus the proof of Proposition 3.1 will follow from the following lemma.

Lemma 4.2. *Let M, N be fixed S -modules. For $k \geq 2$, all continuous natural transformations of the form*

$$\Theta(X) : M \wedge X\langle 0 \rangle \rightarrow N \wedge X\langle 0 \rangle^{\wedge k}$$

are naturally null.

The proof is a bit delicate, and has two rather distinct steps.

The first step is a reduction to Lemma 4.3, below.

Let $F(X) = M \wedge X\langle -1 \rangle$ and $G(X) = N \wedge X\langle -1 \rangle^{\wedge k}$.

Since $(\Sigma^{-1}X)\langle 0 \rangle = \Sigma^{-1}X\langle 1 \rangle$, the natural commutative square

$$\begin{array}{ccc} \Sigma F(\Sigma^{-1}X) & \xrightarrow{\Theta(\Sigma^{-1}X)} & \Sigma G(\Sigma^{-1}X) \\ \downarrow \Delta & & \downarrow \Delta \\ F(X) & \xrightarrow{\Theta(X)} & G(X) \end{array}$$

rewrites as

$$\begin{array}{ccc} M \wedge X\langle 1 \rangle & \xrightarrow{\Theta(\Sigma^{-1}X)} & \Sigma N \wedge (\Sigma^{-1}X\langle 1 \rangle)^{\wedge k} \\ \downarrow & & \downarrow \Delta \\ M \wedge X\langle 0 \rangle & \xrightarrow{\Theta(X)} & N \wedge X\langle 0 \rangle^{\wedge k}. \end{array}$$

The right vertical map factors as the composite

$$\Sigma N \wedge (\Sigma^{-1}X\langle 1 \rangle)^{\wedge k} \rightarrow \Sigma N \wedge (\Sigma^{-1}X\langle 0 \rangle)^{\wedge k} \xrightarrow{\Delta} N \wedge X\langle 0 \rangle^{\wedge k},$$

which is null, as the second map in this composite is the identity map for $N \wedge (\Sigma^{-1}X\langle 0 \rangle)^{\wedge k}$ smashed with the null map $S^1 \xrightarrow{\Delta} S^k$.

We conclude that the composite

$$M \wedge X\langle 1 \rangle \rightarrow M \wedge X\langle 0 \rangle \xrightarrow{\Theta(X)} N \wedge X\langle 0 \rangle^{\wedge k}$$

is naturally null, and thus $\Theta(X)$ naturally factors through a natural transformation of the form

$$M \wedge H\pi_0(X) \xrightarrow{\Psi(X)} N \wedge X\langle 0 \rangle^{\wedge k}.$$

Lemma 4.2 will thus follow from the next lemma.

Lemma 4.3. *Let M, N be fixed S -modules. For $k \geq 2$, all continuous natural transformations of the form*

$$\Psi(X) : M \wedge H\pi_0(X) \rightarrow N \wedge X\langle 0 \rangle^{\wedge k}$$

are naturally null.

To prove this, we let $H(X) = M \wedge H\pi_0(X)$ and, as before, let $G(X) = N \wedge X\langle 0 \rangle^{\wedge k}$, so that our natural transformation has the form

$$\Psi(X) : H(X) \rightarrow G(X).$$

We consider the k th Goodwillie-Taylor approximations G and H . Both functors are finitary in the sense of Goodwillie [Goo03]. From Goodwillie's explicit construction of $p_k F$, if a natural transformation $F(X) \rightarrow F'(X)$ of finitary functors agree on highly connected spectra, then $p_k F(X) \rightarrow p_k F'(X)$ will be an equivalence for all X .

Applying this observation to $N \wedge X\langle 0 \rangle^{\wedge k} \rightarrow N \wedge X^{\wedge k}$, we deduce that $(p_k G)(X) \simeq N \wedge X^{\wedge k}$.

Applying the observation to $M \wedge H\pi_0(X) \rightarrow *$, we see that $(p_k H)(X) \simeq *$.

Now $\Psi(X)$ fits into the diagram

$$\begin{array}{ccccc} H(X) & \xleftarrow{\sim} & H(X\langle 0 \rangle) & \xrightarrow{\sim} & (p_k H)(X\langle 0 \rangle) \simeq * \\ \downarrow \Psi(X) & & \downarrow \Psi(X\langle 0 \rangle) & & \downarrow (p_k \Psi)(X\langle 0 \rangle) \\ G(X) & \xleftarrow{\sim} & G(X\langle 0 \rangle) & \xrightarrow{\sim} & (p_k G)(X\langle 0 \rangle), \end{array}$$

and we conclude that $\Psi(X)$ is null, proving the Lemma 4.3.

5. THE MOD 2 HUREWICZ MAP OF THE CONNECTED COVERS OF BO

In this section we show how Corollary 1.5 allows us to quite easily compute

$$h_* : \pi_*(ko\langle c \rangle) \rightarrow H_*(BO\langle c \rangle; \mathbb{Z}/2)$$

for all $c > 0$. As computing this is equivalent to computing

$$H_n(BO\langle n \rangle; \mathbb{Z}/2) \rightarrow H_n(BO\langle c \rangle; \mathbb{Z}/2)$$

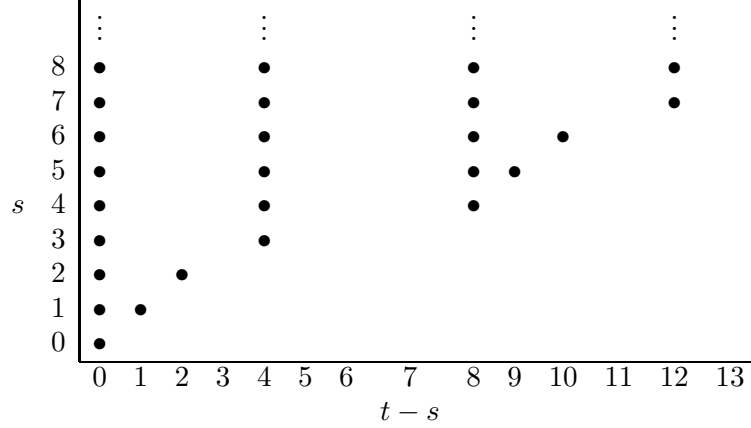
for all $n \geq c$, this result is surely accessible by a close reading of [St63]. However, what we do here involves much less calculation.

It is a standard calculation that the connective real K -theory spectrum ko has mod 2 cohomology as a module over the Steenrod algebra given by

$$H^*(ko; \mathbb{Z}/2) = \mathcal{A}/\mathcal{A}\{Sq^1, Sq^2\} = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$$

Here $\mathcal{A}(1)$ is the sub(Hopf)-algebra generated by Sq^1 and Sq^2 . This leads to the standard picture [Rav86, p.66] of the E_2 -term of the Adams spectral sequence, pictured here as Figure 1, and the conclusion that $E_2 = E_\infty$. (Each \bullet stands for $\mathbb{Z}/2$, as usual.)

It follows quite easily that, when $c \equiv 0, 1, 2, 4 \pmod{8}$, the corresponding Ext chart for $ko\langle c \rangle$ looks identical, except that columns with $t - s < c$ are

FIGURE 1. $\text{Ext}_{\mathcal{A}}^{s,t}(H^*(ko; \mathbb{Z}/2), \mathbb{Z}/2)$

now zero, and Adams filtration s has been decreased so that the bottom class, with degree $t - s = c$, has Adams filtration 0. Again $E_2 = E_\infty$.

For example, the chart for $bo = ko\langle 1 \rangle$ is pictured in Figure 2. Only the four classes labeled with \circ satisfy the inequality $t - s \geq 2^s$. Corollary 1.5 thus tells us that

$$h_* : \pi_n(bo) \rightarrow H_*(BO; \mathbb{Z}/2)$$

can only possibly be nonzero when $n = 1, 2, 4, 8$. This certainly happens: h_* being nonzero in degree n is equivalent to finding a real vector bundle ξ over S^n with top Steifel–Whitney class $w_n(\xi) \neq 0$, and the existence of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and \mathbb{O} allows one to construct the bundles. As the image of h_* lands in the primitives of $H_*(BO; \mathbb{Z}/2)$, a little bit of fiddling reveals that, if $x \in H_1(BO; \mathbb{Z}/2)$ is the nonzero class, then $\text{Im } h_* = \langle x, x^2, x^4, x^8 \rangle$.

(We note that we have reproved the old observation of Milnor [Mil58] that the n sphere admits a bundle ξ with $w_n(\xi) \neq 0$ only when $n = 1, 2, 4, 8$. Milnor uses (then recent) work of Bott closely related to Bott periodicity. Our proof here uses roughly the same ingredients, though rather differently packaged: starting from Bott periodicity it is not too hard to compute $H^*(ko; \mathbb{Z}/2)$ as an \mathcal{A} -module, which is what we need to know.)

Similarly Corollary 1.5 allows us to deduce that

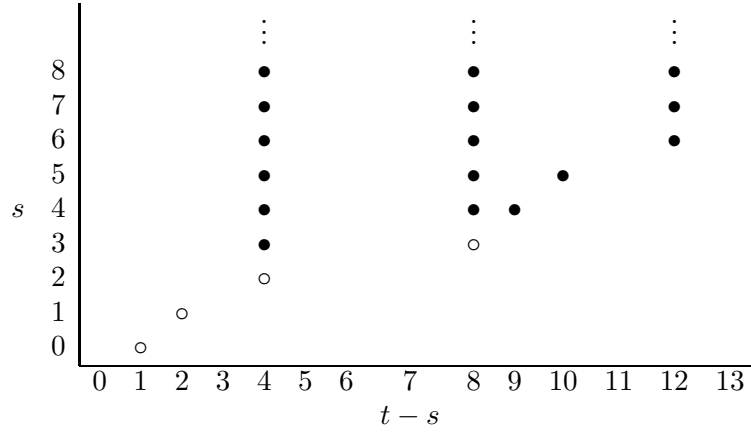
$$\text{Im}\{h_* : \pi_*(bso) \rightarrow H_*(BSO; \mathbb{Z}/2)\} = \langle y, y^2, y^4 \rangle \text{ with } |y| = 2$$

(no surprise, as $BO = BSO \times B\mathbb{Z}/2$ as spaces),

$$\text{Im}\{h_* : \pi_*(bspin) \rightarrow H_*(BSpin; \mathbb{Z}/2)\} = \langle z, z^2 \rangle \text{ with } |z| = 4,$$

and, for all $c \geq 8$, with $c \equiv 0, 1, 2, 4 \pmod{8}$,

$$\text{Im}\{h_* : \pi_*(ko\langle c \rangle) \rightarrow H_*(BO\langle c \rangle; \mathbb{Z}/2)\} = \langle v \rangle \text{ with } |v| = c.$$

FIGURE 2. $\text{Ext}_{\mathcal{A}}^{s,t}(H^*(bo; \mathbb{Z}/2), \mathbb{Z}/2)$

REFERENCES

- [AK02] S. T. Ahearn and N. J. Kuhn, *Product and other fine structure in polynomial resolutions of mapping spaces*, Alg. Geom. Top. **2** (2002), 591–647.
- [B99] M. Bastera, *André–Quillen cohomology of commutative S-algebras*, J. Pure Appl. Alg. **144**(1999), 111–143.
- [BM05] M. Bastera and M. A. Mandell, *Homology and cohomology of E_∞ ring spectra*, Math. Zeit. **249** (2005), 903–944.
- [CM13] S. G. Chadwick and M. A. Mandell, *E_n genera*, preprint, 2013. arXiv 1310.3336.
- [Goo03] T. G. Goodwillie, *Calculus III: Taylor series*, Geometry and Topology **7** (2003), 645–711.
- [Ha09] J.E.Harper, *Homotopy theory of modules over operads in symmetric spectra*, Alg. Geom. Top. **9** (2009), 1637–1680.
- [Ha10] J.E.Harper, *Bar constructions and Quillen homology of modules over operads*, Alg. Geom. Top. **10** (2010), 87–136.
- [HaHe13] J.E.Harper and K.Hess, *Homotopy completion and topological Quillen homology of structured ring spectra*, Geometry and Topology **17** (2013), 1325–1416.
- [K06] N. J. Kuhn, *Localization of André–Quillen–Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. **201** (2006), 318–378.
- [K07] N. J. Kuhn, *Adams filtration and infinite loop spaces*, Oberwolfach Reports **4** (2007), 2703–2705.
- [KM13] N. J. Kuhn and J. B. McCarty, *The mod 2 homology of infinite loopspaces*, Alg. Geom. Top. **13** (2013), 687–745.
- [KP14] N. J. Kuhn and L. Pereira, *Composition products on Topological André–Quillen filtrations*, manuscript in progress, University of Virginia.
- [L88] J.Lannes, *Sur le n-dual du n-ème spectre de Brown–Gitler*, Math. Zeit. **199** (1988), 29–42.
- [LZ83] J. Lannes and S. Zarati, *Invariants de Hopf d’ordre supérieur et suite spectrale d’Adams*, C. R. Acad. Sci. Paris Sér. I Math **296** (1983), 695–698.
- [LZ87] J. Lannes and S. Zarati, *Sur les foncteurs dérivés de la déstabilisation*, Math. Zeit. **194** (1987), 25–59.
- [LZ] J. Lannes and S. Zarati, *Invariants de Hopf d’ordre supérieur et suite spectrale d’Adams*, unpublished manuscript dating from 1983.

- [MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. L.M.S. **82** (2001), 441-512.
- [McMi04] R. McCarthy and V. Minasian, *On triples, operads, and generalized homogeneous functors*, preprint, 2004.
- [Mil58] J. Milnor, *Some consequences of a theorem of Bott*, Ann. Math. **68** (1958), 444-449.
- [Pe13] L. Pereira, *Goodwillie Calculus in the category of algebras over a spectral operad*, manuscript in progress, available on his MIT website.
- [Po11] G. M. L. Powell, *On the derived functors of destabilization at odd primes*, arXiv:1101.0226.
- [Rez09] C. Rezk, *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy, and Applications **11**, 2009, 327-379.
- [Rav86] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics **121**, Academic Press, 1986.
- [Sh04] B. Shipley, *A convenient model category for commutative ring spectra*, Homotopy theory: Relations with algebraic geometry, group cohomology, and algebraic K-theory, A.M.S. Cont. Math. Series **346** (2004), 473-483.
- [St63] R. E. Stong, *Determination of $H^*(BO(k, \dots, \infty), \mathbb{Z}_2)$ and $H^*(BU(k, \dots, \infty), \mathbb{Z}_2)$* , Trans. A.M.S. **107** (1963), 526-544.

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